Application of LP strong duality

I) For a bipartite graph $G=(V, E)$

- A matching is a disjoint subset of edges (Subsets of edges which don't share any common vertex).
- A vertex cover is a subset of nodes that together touch all edges

Lemma 1: The cardinality of any matching is less than or equal to the cardinality of vertex covering. (this doesn't need bipartite)

Pf: Consider $\mathcal{E} \subseteq E$ be a matching. $\mathcal{G} \subseteq G$ be a vertex cover.

Then for each $e \in \mathcal{E} \subseteq E$, we have: $\exists$ one $g \in \mathcal{G}$ which is the vertex of e. And g can't be other vertex of $e' \neq e \in \mathcal{E} \subseteq E$. ☒

Then consider: If G is bipartite, the cardinality of the maximum matching equals the minimum vertex covering. (Konig)

To prove this, we need to introduce:

Def: A matrix A is totally unimodular (TUM) if the determinant of any its square submatrices belongs to $\{0, -1, +1\}$.

Recall our definition for extreme point (non-innerpoint) and vertex (sometimes unique minimizer / maximizer over X general X, sometimes n linearly independent active constraints, $X \subseteq \mathbb{R}^n$ for polyhedron.).

Claim 1: Extreme point $\Leftrightarrow$ Vertex for polyhedron $\{x \mid Ax \le b\}$.

pf: Vertex (def first) $\Rightarrow$ Extreme point $\Rightarrow$ vertex (def latter).

Claim 2: Consider the LP : $\min c^T x$, s.t $Ax \le b \triangleq P$.

If P has a extreme point, then if this LP has a optimal solution, then it admits a optimal extreme point solution.

pf: Notice that: for polyhedron P: (if $\neq \phi$).

① P doesn't contain a line
② P admits one extreme point.
are equivalent ($\Leftrightarrow r(A) = n$).

① $\Rightarrow r(A) = n$, otherwise $N(A) \neq \phi \Rightarrow$ ② (n active l.i. constraints).
② $\Rightarrow$ ① : If P contains a line : $x + \alpha d$, $\alpha \in \mathbb{R}$, $d \neq 0 \in \mathbb{R}^n$, then $Ad = 0$. (consider each component).
$\Rightarrow r(A) < n \Rightarrow$ No extreme point .... $\Rightarrow$ Contradiction

(Rmk for ① $\Rightarrow$ ②: Consider for $x \in P$, let $I = \{ i \in \{1, 2, \dots, m\}, a_i^T x = b_i \}$.
If $i < n$, consider $N(A_I) \neq \phi$, take $d \in N(A_I)$.
Since P doesn't contains a line, there exists λ^*, s.t $\exists$ at least one more constraint is tight for $x + \lambda^* d$ ....).

Now, let v be the optimal value. $P_1 = \{Ax \le b, c^T x = v\}$ is nonempty polyhedron as well. Since P_1 doesn't contain a line, there exists a extreme point of P_1, densted by w.

We now show that w is the extreme point of P. Otherwise, $\exists z_1, z_2 \in P$, s.t. $\lambda z_1 + (1-\lambda) z_2 = w$, $\lambda \in [0,1]$, $z_1, z_2 \in P$.
Consider $c^T z_1$, $c^T z_2$ is enough for contradiction $\square$

Def: A polyhedron $P = \{x \mid Ax \leq b\}$ is integral if all of its extreme points are integral points.

Thm: $P = \{x \mid Ax \leq b\}$, if A is TUM and b is integral, then the polyhedron P is integral as well.

Pf: $\exists$ corresponding $A'x = b'$, $A' \in \mathbb{R}^{n \times n}$ with full rank.

then $x_i = \dfrac{\det(A'_i)}{\det(A')}$, since $\det(A') \neq 0$, $\det(A'_i) \in \mathbb{Z} \Rightarrow$ $\square$

(Rmk: if with no extreme point, obviously)

Now, we can begin our proof!

Proof: Consider the minimum vertex cover problem:

$$\min \sum_{i=1}^{|V|} x_i,$$

s.t $x_i \in \{0,1\}$, $i = 1 \sim |V|$.

$\sum_{i \sim e} x_i \geq 1$, $\forall e \in E$, $i \sim e$ denotes i is the vertex of e.

Let the incidence matrix be $A \in \mathbb{R}^{|V| \times |E|}$
Then, this problem can be rewriten as:

$$\min \sum_{i=1}^{|V|} x_i, \qquad \text{(OMVC)}$$
$$\text{s.t } x_i \in \{0,1\},\ i=1\sim|V|$$
$$A^T x \geq \vec{1},\ x=(x_1,\dots,x_{|V|})^T$$

Relax it, we get

$$\min \sum_{i=1}^{|V|} x_i, \qquad \text{(RMVC).}$$
$$\text{s.t } x_i \geq 0,\ i=1\sim|V|.$$
$$A^T x \geq \vec{1}$$

Consider the dual problem of RMVC.

$$\max \sum_{i=1}^{|E|} \lambda_i, \qquad \text{(RMM)}$$
$$\text{s.t } \lambda_i \geq 0,\ i=1\sim|E|.$$
$$\vec{1} \geq A\lambda,\ \lambda=(\lambda_1,\dots\lambda_{|E|})^T$$

It's similar to the maximum matching problem:

$$\max \sum_{i=1}^{|E|} \lambda_i \qquad \text{(OMM)}$$
$$\text{s.t } \lambda_i \in \{0,1\},\ i=1\sim|E|$$
$$\vec{1} \geq A\lambda,\ \lambda=(\lambda_1,\dots,\lambda_{|E|})^T$$

Notice that: Both OMVC and OMM are feasible and bounded.
Both RMM and RMVC are feasible and bounded.

Since RMM and RMVC are LPs, they can attain their optimal value. and their optimal value are the same.

Now, we just need to establish the comparison between the relaxed problem and unrelaxed problem.

Lemma: incidence matrix A for a bipartite graph is TUM.

Lemma's pf: Prove by the submatrix's size $n \le \min\{|V|, |E|\}$

When $n=1$, obviously.
Assume that when $n \le k$, the result holds. Then consider any subsquarematrix B of A of size $k+1$.

(i) If B has column that has no 1 $\Rightarrow$ Finish, $\det(B)=0$
(ii) If B has column that has exactly one 1 $\Rightarrow$ Finish.
(By expansion of determinant and inductive basis)

(iii) If all B's column has at least two 1. (exactly two 1)
Since the graph is bipartite, we can exchange B's rows, s.t $B = \begin{pmatrix} B' \\ B'' \end{pmatrix}$ and for any column of B, it has exactly one 1 in B' and B'' respectively.

Hence, the sum of B''s row = the sum of B'''s row
$\Rightarrow \det(B) = 0$.

So by induction, we prove our lemma.

Moreover, we can prove:

Lemma: If $A \in \mathbb{R}^{m \times n}$ is TUM, then, consider the standard basis in $\mathbb{R}^m$, denoted by $e_1 \sim e_m$; the standard basis in $\mathbb{R}^n$, denoted by $g_1 \sim g_n$. We have : $(A, e_{i_1}, \cdots, e_{i_k})$, $\{i_1, \cdots, i_k\} \subseteq \{1, 2, \cdots, m\}$ is TUM, $\begin{pmatrix} A \\ g_{i_1}^T \\ \vdots \\ g_{i_w}^T \end{pmatrix}$ is TUM, $\{i_1, \cdots, i_w\} \subseteq \{1, 2, \cdots, n\}$

This is just proved. by expansion step by step.

Now, consider RMM, which can be rewriten as

$$\max \sum_{i=1}^{|E|} \lambda_i$$
$$\text{s.t} \begin{pmatrix} A \\ -I \end{pmatrix} \lambda \leq \begin{pmatrix} \vec{1} \\ 0 \end{pmatrix}.$$

Since $\begin{pmatrix} A \\ -I \end{pmatrix}$ is TUM, $\begin{pmatrix} \vec{1} \\ 0 \end{pmatrix}$ is integral, the extreme point of the polyhedron $\begin{pmatrix} A \\ -I \end{pmatrix} \lambda \leq \begin{pmatrix} \vec{1} \\ 0 \end{pmatrix}$ is integral.

And since RMM has an optimal solution and a extreme point $\lambda = \vec{0}$, RMM has an optimal extreme point solution λ^*.

Then λ^* is integral and each of its component can't be less than 0 or bigger than 1.

$\Rightarrow \lambda^*$ is also the solution of OMM.

Hence, the solution of OMM exists and equals to the solution of RMM.

Similarly for OMVC and RMVC, by strong duality we get the result.